

Renormalization group analysis of a turbulent compressible fluid near $d = 4$: Crossover between local and non-local scaling regimes.

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Abstract. We study scaling properties of the model of fully developed turbulence for a compressible fluid, based on the stochastic Navier-Stokes equation, by means of the field theoretic renormalization group (RG). The scaling properties in this approach are related to fixed points of the RG equation. Here we study a possible existence of other scaling regimes and an opportunity of a crossover between them. This may take place in some other space dimensions, particularly at $d = 4$. A new regime may there arise and then by continuity moves into $d = 3$. Our calculations have shown that there really exists an additional fixed point, that may govern scaling behaviour.

1 Introduction

A majority of works on fully developed turbulence is concerned with an incompressible fluid. The renormalization group approach to such problems has been successful in verifying Kolmogorov scaling and provides an efficient tool for a calculation of universal quantities. However, a similar treatment has been only scarcely applied to compressible fluids. In this paper we present an application of the field theoretic renormalization group (RG) onto the scaling regimes of a compressible fluid, whose behavior is governed by a proper generalization of stochastic Navier Stokes equation [1]. Similar models of compressible fluid were considered in [2–4]. In [2] the phenomenological corrections to the Kolmogorov spectrum were verified in the framework of the skeleton equations for consistency, while the model, considered in [3], appears to be in fact unrenormalizable. All these papers show us a necessity of the further investigations of compressibility.

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Following [5], we employ double expansion scheme. Here the formal expansion parameters are y , which describes the scaling behavior of a random force, and $\varepsilon = 4 - d$, i.e., a deviation from the dimension of space $d = 4$.

2 Description of the model

The Navier-Stokes equation for a compressible fluid can be written in the following form:

$$\rho[\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v}] = \nu_0[\nabla^2 \mathbf{v} - \nabla(\nabla \cdot \mathbf{v})] + \mu_0 \nabla(\nabla \cdot \mathbf{v}) - \nabla p + \mathbf{f}, \quad (1)$$

where ρ is the fluid density, \mathbf{v} is the velocity field, ∂_t is a time derivative $\partial/\partial t$, ∇^2 is the Laplace operator, ν_0 and μ_0 are molecular viscosity coefficients, p is pressure field, and \mathbf{f} is an external field per unit mass. The model must be augmented by two additional equations, namely a continuity equation and an equation of state between deviations δp and $\delta \rho$ from the equilibrium values. They read

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0; \quad (2a)$$

$$\delta p = c_0^2 \delta \rho. \quad (2b)$$

In order to obtain the renormalizable field theoretic model expression (1) is divided by ρ , and fluctuations in viscous terms are neglected [6]. Further, by using the continuity equation and the equation of state (2), the problem can be recasted in terms of two coupled equations:

$$(\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} = \nu_0[\nabla^2 \mathbf{v} - \nabla(\nabla \cdot \mathbf{v})] + \mu_0 \nabla(\nabla \cdot \mathbf{v}) - \nabla \phi + \mathbf{f}; \quad (3a)$$

$$(\partial_t + \mathbf{v} \cdot \nabla) \phi = -c_0^2 (\nabla \cdot \mathbf{v}). \quad (3b)$$

Here ϕ is related to the density fluctuations via the relation $\phi = c_0^2 \ln(\rho/\bar{\rho})$. Parameter c_0 is an adiabatic speed of sound, $\bar{\rho}$ denotes the mean value of ρ .

The turbulence is modeled by an external force – it is assumed to be a random variable, which should mimic the input of the energy into the system from the outer large scale L . Its precise form is believed to be unimportant and is usually considered to be a random Gaussian variable with zero mean and correlator

$$\langle f_i(t, \mathbf{x}) f_j(t', \mathbf{x}') \rangle = \frac{\delta(t - t')}{(2\pi)^d} \int_{k>m} d^d \mathbf{k} D_{ij}(\mathbf{k}) e^{ik \cdot (\mathbf{x} - \mathbf{x}')}, \quad \text{where} \quad (4a)$$

$$D_{ij}(\mathbf{k}) = g_{10} \nu_0^3 k^{4-d-y} \left\{ P_{ij}(\mathbf{k}) + \alpha Q_{ij}(\mathbf{k}) \right\}. \quad (4b)$$

Here d is the space dimension, $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$ and $Q_{ij}(\mathbf{k}) = k_i k_j / k^2$ are the transverse and longitudinal projectors, $k = |\mathbf{k}|$, a parameter $m = L^{-1}$ provides an infrared (IR) cutoff, amplitude α is a free parameter, an exponent y plays a role of a formally small expansion parameter, and g_{10} is a coupling constant; Dirac delta function ensures Galilean invariance [7].

3 Field theoretic formulation of the model

According to the general theorem [8, 9], the stochastic problem is equivalent to the field theoretic model with a doubled set of fields $\tilde{\psi}, \psi$ and de Dominicis-Janssen action functional, written in a compact form as

$$\mathcal{S}(\varphi) = \frac{v'_i D_{ik}^f v'_k}{2} + v'_i \left\{ -\partial_i v_i - v_j \partial_j v_i + \nu_0 [\delta_{ik} \partial^2 - \partial_i \partial_k] v_k + u_0 \nu_0 \partial_i \partial_k v_k - \partial_i \phi \right\}$$

$$+ \phi' [-\partial_t \phi + v_j \partial_j \phi + v_0 v_0 \partial^2 \phi - c_o^2 (\partial_i v_i)]. \quad (5)$$

Here we have employed a condensed notation, in which integrals over the spatial variable \mathbf{x} and the time variable t , as well as summation over repeated indices, are implicitly assumed. The action (5) is amenable to the standard methods of the quantum field theory, such as the Feynman diagrammatic technique and the renormalization group procedure.

In a standard approach, if we apply quantum field methods to the stochastic differential equations, the space dimension d plays a passive role and an actual perturbative parameter is y ; for more details see the monographs [7, 8]. Our approach closely follows the analysis of the incompressible Navier-Stokes equation near space dimension $d = 2$ (see [5, 10–12]). In this case three additional divergences appear in the Green's function $v'v'$. They can be absorbed by a suitable local counterterm $v'_i \nabla^2 v'_i$, and a regular expansion in both y and $\varepsilon' = d - 2$ was constructed. Up to now the present model (5) has been investigated at the fixed space dimension $d = 3$, for which the action (5) contains all terms that can be generated during the renormalization procedure [1, 13–15]. However, using the dimensional analysis it can be shown that at $d = 4$ there appears an additional divergence, also in the Green's function $v'v'$. Therefore, to keep the model renormalizable at $d = 4$ the kernel function in (4) has to be generalized to the following form:

$$D_{ij}(\mathbf{k}) \rightarrow g_{10} v_0^3 k^{4-d-y} \left\{ P_{ij}(\mathbf{k}) + \alpha Q_{ij}(\mathbf{k}) \right\} + g_{20} v_0^3 \delta_{ij}, \quad (6)$$

where the new term on the right hand side absorbs divergent contributions from $v'v'$. In contrast to [5] no momentum dependence is needed.

4 Feynman diagrammatic technique

The perturbation theory of the model can be expressed in the standard Feynman diagrammatic expansion [8, 16]. Bare propagators are read off from the inverse matrix of the Gaussian (free) part of the action functional, while the nonlinear part of the differential equation defines the interaction vertices. Their graphical representation is depicted in Fig. 1. Explicit expressions of propagators in frequency-momentum representation can be found, e.g., in [1], and they are right for actual calculations.

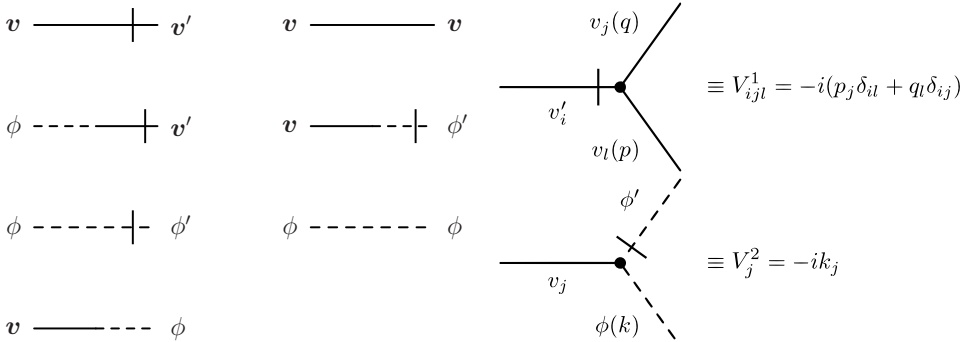


Figure 1. Graphical representation of the bare propagators and interaction vertices in the model (5)

The ultraviolet (UV) renormalizability is very efficiently revealed by an analysis of the 1-irreducible Green's functions. Corresponding generating functional can be written in the form

$$\Gamma(\varphi) = \mathcal{S}(\varphi) + \widetilde{\Gamma}(\varphi), \quad (7)$$

where for the functional arguments we have used the same symbols $\varphi = \{v, v', \phi, \phi'\}$ as for the corresponding random fields [8]; $\mathcal{S}(\varphi)$ is the action functional (5) and $\widetilde{\Gamma}(\varphi)$ is the sum of all the 1-irreducible diagrams with loops [8]. As it has been shown in [1] and discussed in [14], the model (5) – (6) is invariant with respect to the Galilean symmetry, which results to the UV finiteness of the two Green's functions: $v_i \partial_i v_i$ and $v'_i (v_j \partial_j) v_i$. We have carried on the perturbative analysis in the one-loop order, consequently the expressions for the 1-irreducible Green's functions, which requires UV renormalization, can be formally written in the following way:

$$\Gamma_{vv} = i\omega - (\delta_{ij} p^2 - p_i p_j) Z_1 v - p_i p_j Z_2 uv + \text{diagram}, \quad (8)$$

$$\Gamma_{\phi\phi'} = i\omega - p^2 Z_3 v v + \text{diagram}, \quad (9)$$

$$\Gamma_{v'\phi} = -i Z_4 p_i + \text{diagram}, \quad (10)$$

$$\Gamma_{\phi'v} = -i Z_5 p_i c^2 + \text{diagram} + \text{diagram} + \text{diagram}, \quad (11)$$

$$\Gamma_{v'v'} = g_1 v^3 p^{4-d-y} \{P_{ij}(\mathbf{p}) + \alpha Q_{ij}(\mathbf{p})\} + g_2 v^3 \delta_{ij} Z_6 + \frac{1}{2} \text{diagram}, \quad (12)$$

where \mathbf{p} always represents a corresponding external momentum. A factor 1/2 in front of the diagram in (12) denotes a symmetry coefficient of the given graph. Collecting all the mentioned facts and taking into account that non-local terms should not be renormalized, it is straightforward to show that the theory is UV renormalizable. From the direct comparison of the relations between renormalized parameters it follows that

$$\begin{aligned} Z_v &= Z_1, & Z_{g_1} &= Z_1^{-3}, & Z_u &= Z_2 Z_1^{-1}, & Z_\phi &= Z_4, \\ Z_{\phi'} &= Z_4^{-1}, & Z_v &= Z_3 Z_1^{-1}, & Z_c &= (Z_4 Z_5)^{1/2}, & Z_{g_2} &= Z_6 Z_1^{-3}. \end{aligned} \quad (13)$$

Employing dimensional regularization within minimal subtraction scheme (MS) [17] the renormalization constants can be calculated and the UV divergences manifests themselves in pole terms in y and $\varepsilon = 4 - d$. In higher loops pole terms in form of general linear combination in $ay + b\varepsilon$ may appear.

5 UV renormalization of the model and fixed points

The large scale behavior with respect to spatial and time scales is governed by the IR attractive stable fixed point $g^* \equiv \{g_1^*, g_2^*, u^*, v^*\}$. Here and henceforth the asterisk refers to a coordinate of the fixed point (FP). Their coordinates are determined from the relations [8, 16]

$$\beta_{g_1}(g^*) = \beta_{g_2}(g^*) = \beta_u(g^*) = \beta_v(g^*) = 0, \quad (14)$$

where $\beta_x = \widetilde{D}_\mu x$ for any variable x , and differential operator \widetilde{D}_μ denotes operation $\mu \partial_\mu$ at fixed bare parameters $\{g_{10}, g_{20}, u_0, v_0, \nu_0, \alpha_0, c_0\}$; μ is the “reference mass” (additional free parameter of the renormalized theory) in the MS renormalization scheme. The eigenvalues of the matrix of first derivatives

$\Omega_{ij} \equiv \partial\beta_i/\partial g_j$, where $i, j \in \{g_1, g_2, u, v\}$, determine whether the given FP is IR stable or not. Points with all positive eigenvalues are candidates for macroscopic regimes and in principle can be observed experimentally. An explicit forms of the β -functions are

$$\beta_{g_1} = g_1(-y - \gamma_{g_1}), \quad \beta_{g_2} = g_2(-\varepsilon - \gamma_{g_2}), \quad \beta_u = -u\gamma_u, \quad \beta_v = -v\gamma_v, \quad (15)$$

where $\gamma_x = \widetilde{D}_\mu \ln Z_x$ are the anomalous dimensions [8]. A direct analysis of the system of equations (14) reveals the existence of three IR stable fixed points: FPI, FPII and FPIII. FPI is the free (Gaussian) fixed point, for which all interactions are irrelevant and no scaling and universality is expected. Its coordinates are

$$g_1^* = 0, \quad g_2^* = 0, \quad \text{whereas } u^* \text{ and } v^* \text{ are undetermined.} \quad (16)$$

The corresponding eigenvalues of the matrix Ω are

$$\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = -\varepsilon, \quad \lambda_4 = -y. \quad (17)$$

Though trivial, this point is necessary for the correct use of perturbative renormalization group.

Further, there is a local fixed point FPII, for which the charge g_2 attains a non-zero value, and corresponding coordinates are

$$g_1^* = 0, \quad g_2^* = \frac{8\varepsilon}{3}, \quad u^* = 1, \quad v^* = 1. \quad (18)$$

The eigenvalues of the matrix Ω are

$$\lambda_1 = \frac{7\varepsilon}{18}, \quad \lambda_2 = \frac{5\varepsilon}{6}, \quad \lambda_3 = \varepsilon, \quad \lambda_4 = \frac{3\varepsilon - 2y}{2}. \quad (19)$$

For the last fixed point, FPIII, both non-local and local parts of the random force are relevant:

$$g_1^* = \frac{16y(2y - 3\varepsilon)}{9(y(2 + \alpha) - 3\varepsilon)}, \quad g_2^* = \frac{16\alpha y^2}{9(y(2 + \alpha) - 3\varepsilon)}, \quad u^* = 1, \quad v^* = 1; \quad (20)$$

the required eigenvalues are

$$\lambda_1 = \frac{y[2y(10\alpha + 11) - 3\varepsilon(3\alpha + 11)]}{54[y(2 + \alpha) - 3\varepsilon]}, \quad \lambda_2 = \frac{y[2y(2\alpha + 3) - \varepsilon(\alpha + 9)]}{6[y(\alpha + 2) - 3\varepsilon]}, \quad \lambda_{3,4} = \frac{A \pm \sqrt{B}}{C}, \quad (21)$$

where A, B and C are given by the following expressions:

$$A = -27\varepsilon^3 + 9(9 + \alpha)\varepsilon^2y - 9(8 + 3\alpha)\varepsilon y^2 + 2y^3(\alpha^2 + 7\alpha + 10); \quad (22)$$

$$B = [-3\varepsilon + (2 + \alpha)y]^2[81\varepsilon^4 - 54\varepsilon^3y - 9(3 + 20\alpha)\varepsilon^2y^2 + 12(1 + 17\alpha + 3\alpha^2)\varepsilon y^3 - 4(-1 + 14\alpha + 5\alpha^2)y^4]; \quad (23)$$

$$C = 6[-3\varepsilon + (2 + \alpha)y]^2. \quad (24)$$

From the physical interpretation of the kernel function (4) it follows that the charges g_1^* and g_2^* can not attain negative values. Using this fact together with an explicit form of the eigenvalues $\lambda_1 \dots \lambda_4$ it can be shown, that the point FPIII is stable for $y > 0$ and $y > 3\varepsilon/2$. Note, that the crossover between two nontrivial points happens along the line $y = 3\varepsilon/2$, which is in accordance with [18].

6 Conclusion

In this paper the compressible extension of the stochastic Navier Stokes equation has been studied using the field theoretic approach. Crucial points of the Feynman diagrammatic technique and perturbative renormalization group have been discussed. One loop approximation provides that, depending

of the exponent y and deviation from the dimension of x space $\varepsilon = 4 - d$, the model possesses three stable fixed points in the IR region (i.e., three possible scaling regimes) – trivial (Gaussian, FPI), local (FPII) and nonlocal (FPIII).

This shows us, that the simple analysis around $d = 3$, which indicates existence of only one nontrivial fixed point [1], is incomplete in this case.

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